20/10/22 MATH 4030 Tutorial.

Reminders:

Assignment 3 due Friday night 11:59pm.

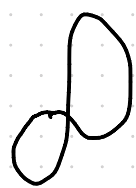
Today's topic: Isoperimetric Inequality — not covered for final exam. End of chapter 1 of do Carmo.

Question: Of all curves in the plane with length l, which one bounds the largest area?

- example of a "global" question
- Here, going to restrict our attention to closed curves. C is closed if

$$\alpha : [a,b] \to \mathbb{R}^2 \text{ with } \alpha(a) = \alpha(b),\ \alpha'(a) = \alpha'(b),\ \ldots$$

- Simple: no self-intersections, i.e. for all $t_1 \neq t_2 \in (a,b)$, $\alpha(t_1) \neq \alpha(t_2)$.



Jordan Curve Thm: Simple closed curves split plane into two regions.

Thm (Isoperimetric Inequality): Let C be a simple closed curve in the plane with length l. Let A be the area of the region bounded by C. Then

$$l^2 - 4\pi A \geq 0$$

with equality if and only if C is a circle.

Lemma: Let A be the area of the region bounded by a positively oriented simple closed curve C: $\alpha(t) = (x(t), y(t))$, $t \in [a,b]$, then

$$A = -\int_a^b y(t)\, x'(t)\, dt = \int_a^b x(t)\, y'(t)\, dt = \frac{1}{2}\int_a^b (xy' - yx')\, dt.$$

Pf: 1st formula: special case of Green's Thm.

2nd equality: $\displaystyle\int_a^b xy'\,dt = \int_a^b (xy)'\,dt - \int_a^b x'y\,dt = \overset{0}{\cancel{(xy(b) - xy(a))}} - \int_a^b x'y\,dt$

$$= -\int_a^b x'y\,dt.$$

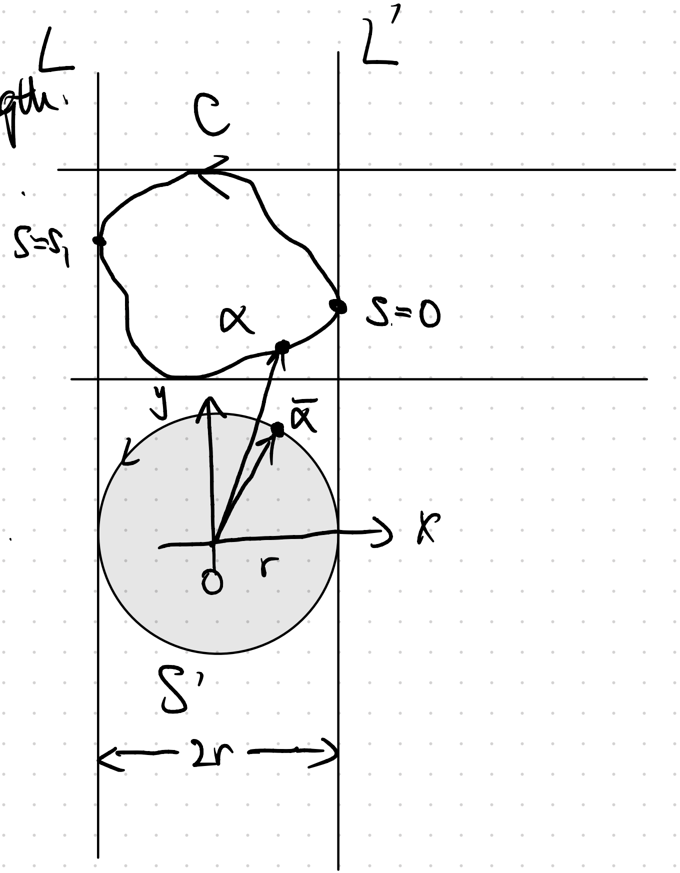
3^{rd} formula: average of first two.

Pf of Thm: (Schmidt '39):

$\alpha(s) = (x(s), y(s))$, s ~ arc-length, $s \in [0, l]$.

$\bar{\alpha}(s) = (\bar{x}(s), \bar{y}(s))$

$= (x(s), \bar{y}(s))$



$$A + \bar{A} = A + \pi r^2$$
$$= \int_0^l x y' \, ds - \int_0^l \bar{y} x' \, ds.$$

$$A+\bar{A} = \int_0^l (xy' - \bar{y}x')\,ds \le \int_0^l |xy' - \bar{y}x'|\,ds$$

$v = (x, \bar{y})$

$w = (y', -x')$

Cauchy-Schwarz inequality states

$$|v \cdot w|^2 \le |v|^2 |w|^2.$$

with equality iff $w = \lambda v$ for some $\lambda \in \mathbb{R}$.

$$v \cdot w = xy' - \bar{y}x', \quad \le \int_0^l \sqrt{x^2+\bar{y}^2} \cdot \underbrace{\sqrt{(y')^2+(x')^2}}_{\tau = |\tau| = 1}\,ds. = \int_0^l \underbrace{\sqrt{\bar{x}^2+\bar{y}^2}}_{\tau = |\bar{x}| = r}\,ds = lr.$$

$A+\hat{A} \le lr.$ Now by Arithmetic-Mean Geometric-Mean inequality

$$\sqrt{A}\sqrt{\bar{A}} = \sqrt{A}\sqrt{\pi r^2} \le \frac{1}{2}(A+\pi r^2) \le \frac{1}{2}lr. \quad \text{equality holds iff } A = \pi r^2.$$

$$\Rightarrow 4A\pi r^2 \le l^2 r^2 \Rightarrow l^2 - 4\pi A \ge 0.$$

Equality Case: Suppose $\ell^2 - 4\pi A = 0$. $\Rightarrow A = \pi r^2 \Rightarrow \ell = 2\pi r$.

Equality case in C-S ε $\Rightarrow v = \lambda w$. for some $\lambda \in \mathbb{R}$.

$$(\bar{x}, \bar{y}) = \lambda(y', -x'). \qquad |v| = |\lambda||w|.$$

$$\lambda = \frac{\bar{x}}{y'} = \frac{-\bar{y}}{x'} = \frac{\sqrt{\bar{x}^2+\bar{y}^2}}{\underbrace{\sqrt{(x')^2+(y')^2}}_{1}} = \pm r. \quad \Rightarrow \bar{x} = \pm r y'.$$

$$\Rightarrow \bar{y} = \pm r x'$$

Then $|x(s)|^2 = \bar{x}^2 + \bar{y}^2 = r^2((x')^2 + (y')^2) = r^2$. $\Rightarrow$ C is the circle. //

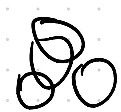
Steiner c1840s gave geometric proofs - non-rigorous b/c assumed existence of maximum.

Weierstrass c 1870-1880 gave rigorous proof using calculus of variations.

Hurwitz 1902. - gave a proof using Fourier analysis.

" "Integral geometry" based proof.

"geometric probability"



Buffon's Needle Problem: $\rightarrow O(\pi)$.